Music, mathematics and language: chronicles from the Oumupo sandbox

M. Andreatta, M. Granger, T. Johnson and V. Villenave

June 30, 2017

Abstract

What is music... if not, at the end of the day, an accessible, fun and expressive way to engage with mathematics and language? **Oumupo** (*Ouvroir de Musique Potentielle*, a Workshop for Potential Music) is a group where musicians and theorists can explore this open question through different exercises and experiments.

A potential history of potential creation

In 1960, François Le Lionnais and Raymond Queneau founded a collective comprised of writers and mathematicians, whose singular objective was to reinvigorate literary forms. This collective, called **Oulipo** (*Ouvroir de Littérature Potentielle*, or Workshop for Potential Literature), has been the source of audacious and thought-provoking works such as Queneau's *Hundred Thousand Billion Poems* (1961) and Perec's *A Void* (1969); it continues to exist today, and has been supplemented for the past two decades with a separate, though perhaps equally interesting, online community: the Oulipo mailing list.

From Oulipo's very inception, François Le Lionnais also imagined extending the scope of its approach to various other disciplines. In this spirit, he founded several other Ou*x*po workshops: Ou*pein*po (as in 'Potential Painting'), Ou*math*po, Ou*ciné*po... Music was very much included in Le Lionnais' scope, and several Ou*mu*po groups did coexist (in a rather informal way) over the next few decades. It was not until 2011, however, that an actual, established **Ouvroir de Musique Potentielle** did take place; while still resolutely part of this tradition, it remains an active and constantly evolving group.

Ou*x*po groups are concerned with inventing new structures and forms: their experiments may however vary in scale, from macroscopic structural constraints to a more minute level where they may be led to reinvent the very language through which their art is expressed. For example, a writer that forbids the use of certain letters or patterns of letters is then forced to express herself within a restricted subset of the normally-available lexicon, thereby needing to resort to unknown words, unknown or archaic formulations that may surprise the reader. The words take on a quality in and of themselves as *signifiers* in addition to, or even before, being understood as *signified*.

In this sense, Oumupo is not only brought to reevaluate purely musical constructs (harmony, rhythm, melody, timbre) but also to draw parallels between these devices and other areas of study: texts or words, graphic arts, but also mathematical objects — automata, geometry, and numbers. As it happens, music often is an excellent and accessible way to wrap one's head around abstract notions or challenging mathematical problems — and in a reciprocal manner, mathematics may be a constant source of inspiration in the elaboration of new musical processes. The next sections aim to cursorily introduce a few of these avenues.[1]

Ou$x$pian games and inventions

As early as 1962, Le Lionnais laid out some useful ground rules in a text later known as the *First Oulipo Manifesto*, detailing the group's orientation and modus operandi[2]. Interestingly, he made the point that Oulipo was not a literary movement, nor had it any intention of turning into one — in fact, the very notion of "inspiration" (and with it any sense of ethereal poetry, randomness or surrealist-like dream associations) was repeatedly rejected. In its place, Oulipo members were aiming to conceive a seemingly-scientific method from the ground up (remotely inspired by the Bourbaki group that, at the time, was in full trend in the French mathematical field), deriving new "potential" structures and constraints from existing works (even going as far back as classical antiquity and medieval literature), as well as from *not-yet-existing* works.

Interweaving the study of past artistic works with the creation of entirely new material (the "analytical" and the "synthetical" approach) remains one of Oulipo's most decisive choices — one that resonated with both artistic audacity (the group co-opted Marcel Duchamp as one of its very first corresponding members, which he gladly accepted) and the modernity of a society yet to come: it would be decades before "remix" even became a word.

This two-pronged approach (dubbed respectively "anoulipism" and "synthoulipism") is still found today in every Ou$x$po group in existence; it has grown much richer precisely due to the increased diversity in "potential" disciplines and languages. For example, the *pictée* ("pictation"), a game originally proposed by Ou*pein*po and therefore relevant only to graphical expression (the game consists of describing an existing artwork to somebody who then produces a new drawing based only on the given clues), was adapted by Oulipo as the *textée* ("textation"), and more recently by Oumupo as the *chansonnée* ("singtation"), where it becomes even more amusing since one has to account for both the lyrics and the music.

Many literary forms can be straightforwardly transposed to the musical realm. A foremost example is the palindrome: as a literary (and possibly religious) practice, it dates back at least to ancient Rome, but it was applied to music at least as early as the 14$^{\text{th}}$ century – as a matter of fact, collecting and reviewing existing examples of musical palindromes and *anacycli* (as well as writing new ones) has been one of the tasks that Oumupo member Valentin Villenave has set out to accomplish.

Not all literary constraints are as easily converted into musical exercises. At least as old (if not older) as the palindrome, is the well-known "lipogram": a text written without using one or more letters of the given alphabet. The obvious musical equivalent would be the "liponote", whereby one is prevented from using a specific pitch (or a degree of the scale) — preferably one of the most essential, in the same way that there is not much point to a lipogram in X or Z. But unlike the lipogram, the liponote quickly proves to be of limited interest. Not only does it hardly make sense in atonal music, but even in classical music one can write around the missing pitch

[1] Some of these examples have been presented in the French popular science magazine *Maths Langages Express*, intended for a large public audience (CIJM, 2017).

[2] Oulipo has had three consecutive manifestos, the first two of which are included in a 1973 book signed collectively (Oulipo, 1973); the third one was drafted during the 1970s but only published much more recently, nearly half a century after the group's foundation (Oulipo, 2009).

through modulations, substitutions and ornaments. This led Valentin Villenave to suggest another Oumupian exercise: the "lipoval", where a specific *interval* is banned, both between simultaneous and consecutive notes (even accounting for added notes in-between). Depending on its scope (which can be of any length, from a single bar to the whole piece), this constraint may become more challenging (and, much like the lipogram, may remain unnoticed by both performers and listeners).

The conquest of Niagara: a radical approach to sound compression!

A peculiar example of "anoumupism" (the musical equivalent of "anoulipism" as explained above) has been proposed by one of our contributors, Martin Granger, with regard to a problem of the utmost importance in the digital age — namely, music compression.

The entire history of musical compression can be summed up in a single phrase: putting as much information as possible into the smallest space possible. But why on Earth would anyone want to fill a hard drive with ten years of music? No one in their right minds, nowadays, would ever spend ten years of her life actually listening to the stuff? What an outrageously futile (not to mention, financially unrewarding) activity! Therefore, the time has come to provide the modern citizen with innovative compression solutions that actually correspond to people's needs. To name but a few of these:

- The **acceleration** of recorded music is a compression method as old as the record itself. In playing a 33 rotations-per-minute record at a speed of 45 RPM, one achieves a time reduction of 36% (an acceleration of 136% is even possible at 78 RPM). Of course, analog time-compression pushes the original frequencies higher, thereby causing the music to be transposed. Which is why only certain pieces lend themselves to this treatment — for example, "interesting" results are obtained with Erik Satie's third *Gymnopédie*, entitled *slow and grave*.

- The **removal of redundant patterns** is a very efficient method, that does not induce any loss of information. Indeed, many composers (undoubtedly out of complacency or sheer laziness) routinely engage in self-plagiarism and shamelessly re-use entire fragments already written: repeat signs, repeated measures, ostinato patterns... Removing all of these is straightforward enough. Take, for instance, the famous first prelude (BWV 846) in C major in J.S. Bach's *Well-Tempered Clavier* (see figure 1). Every bar is made of two identical 8-note patterns, of which the last three notes are actually played twice. An efficient algorithm would first remove one half of each measure, then the last three notes of the remainder. We are left with only 5 notes per measure instead of 16 notes, thus amounting to a 69% temporal gain!

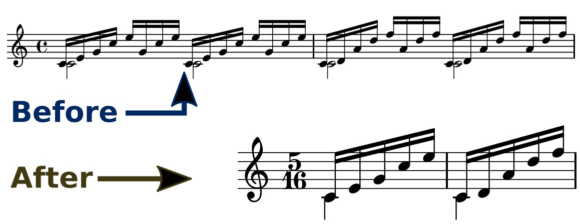

Figure 1: J.S. Bach finally streamlined and enhanced for optimal efficiency: a long-overdue revamping, courtesy of Oumupo!

- The more difficult method of **superposition** requires searching similar passages in one or several works and playing them simultaneously. An interesting research subject is the "rhythm changes" (nicknamed *anatole* in French), the chord progression underlying many jazz standards and popular music. By superposing all of the works constructed using this sequence, one could gain quite a lot of time!

- Last but not least, **random superposition** allows for extremely efficient temporal gains, at the price of dramatically reducing the music's signal-to-noise ratio. For example, Heinrich Ignaz Franz von Biber's *La Battalia a 10* (1673) superposes eight popular melodies in different tonalities and tempos to evoke the theme of drunken soldiers. That 8/1 compression rate allows for an 87.5% reduction in time. In a similar vein, *Folk Music* by Zygmunt Krause (1972) offers a superposition of twenty folk melodies — a whopping 95%!

  Pushed to its extreme, this logic would allow one to listen to *all possible music* at the same time. Also known as "white noise," this is compression at its limit, as the quantity of musical information tends towards zero.

In other words, let us be proactive for once, and skip directly to the ultimate compression: the inevitable day where listening to music will be indistinguishable from hearing the sound of Niagara Falls.

Coding and counting

Given the number of mathematicians that were (and still are) involved with Oulipo, one can but wonder why its sister group Oumupo did not take off immediately when a few informal meetings took place between François Le Lionnais, Michel Philippot, Pierre Barbaud and a few others in the early 1970s. Indeed, music is, in and of itself, a mathematical language, as Leibniz famously posited in his 1712 letter to a young Goldbach: *Musica est exercitium arithmeticæ occultum nescientis se numerare animi* ("music is a hidden exercise in arithmetics, where the mind is unaware it is counting").

Any musician can attest how ubiquitous numbers are in the musical field, in many different forms. To demonstrate their versatility, one may translate a given numerical sequence through various methods; in figures 2 to 10, we will be using the first decimals of the number π as a example.

- **Rhythms**, which include the duration of notes and rests, but as well the number of repetitions of a note or a group of notes (figure 2). This even extends to the quantity of events within a single structural unit (for example, the number of notes in a measure, figure 3).

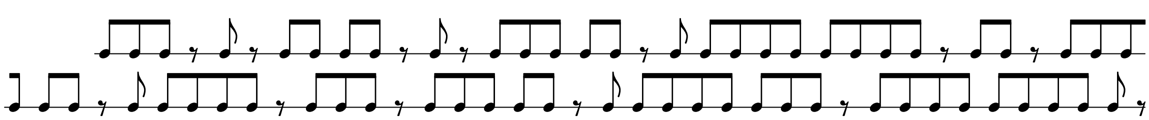

Figure 2: π's first decimals in base 10 (3.141592653589) as rhythmic patterns of various lengths. . .

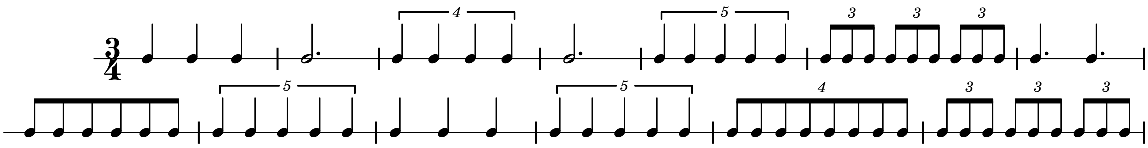

Figure 3: . . . And as quantities of events per measure, with variable speeds.

- **Absolute pitch**, as the degrees of the diatonic scale (figure 4), that are often numbered with Roman Numerals or designated with letters of the alphabet. Since the 20$^{\text{th}}$ century, it is even possible to designate a pitch by its periodic frequency in hertz.

Figure 4: π's first decimals expressed through scale degrees. The zero is rendered by a rest.

- **Relative pitch**, or the interval between a pitch and another one, played either in a sequential or simultaneous manner. By counting the number of semitones between two notes, it is possible to establish detailed metrics of chords and harmonies (figure 5), although base 10 may not be the most appropriate system here, as we will see. In a similar line of thought, the *Tonnetz* (described in a later section) is an original and interesting tool.

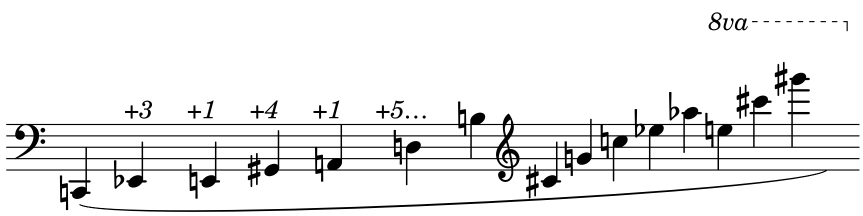

Figure 5: π's first decimals used as cumulative intervals between notes.

In many cases, expressing a number or a mathematical operation through musical elements requires to first find the most appropriate **numeral base**. While we are used to manipulating numbers in a base 10 scheme, musical organization tends to be conceived in entirely different ways.

- Beats and measures are often counted in units that are 2^x, leading to a measurement system in base 4 (the four-bar "hypermeasure" that is the bread-and-butter of European Common Practice musical bar structure) or base 8 (the "count to eight" more common for dancers). At a lower level, sixteenth notes are prevalent in many Western musical traditions: figure 6 is an attempt to express the number π in a manner more suitable to such musicians.

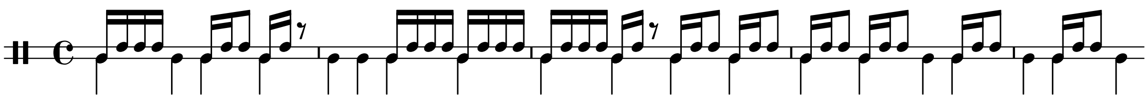

Figure 6: The first digits of π's expansion in base 4 (3.021003331222202020), expressed through pure rhythm. Patterns now occur in a way that becomes much easier to identify by ear.

- The diatonic scale, an omnipresent seven-note construct in many different musical traditions (to begin with the Western one), would therefore require to count in base 7. Pentatonic scales, which comprise five notes, are found in many Eastern cultures: figure 7 is generated again from the number π, in base 5.

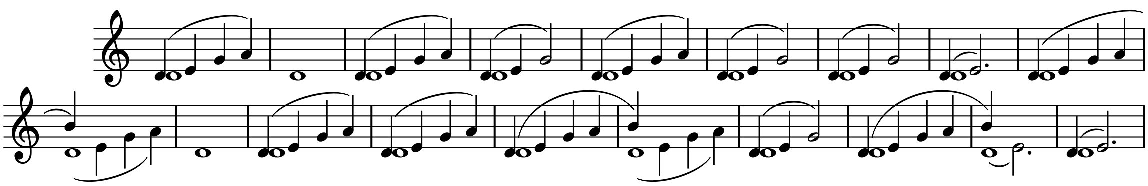

Figure 7: The first digits of π's expansion in base 5 (3.032322143033432411), played along a pentatonic scale. The specific color of this scale adds to the fascinating, hypnotic effect of the repetitive counting pattern.

- The chromatic scale, used since the time of Pythagorean theorists, requires the partitioning of a musical octave in a twelve-semitone scale. Today, this scale is a strict geometric series (where each note is incremented by a factor of $2^{1/12}$), at the basis of the traditional 12 equal temperament system (henceforth 12-ET system).

Figure 8: The first digits of π's expansion in base 12 (3.184809493B918664573A6211BB151551A05729290A780)...

While the ubiquity of 12-ET makes base 12 an obvious choice for turning numbers into notes, it is nevertheless possible to cheat through various means of musical cunning. For example, a ten-note set can be derived from the twelve-note space by omitting two notes from the chromatic collection, i.e., by removing them altogether, or by making them into pedal or drone tones. Compare, for example, figures 8 and 9 where the number π is expressed in proper base 12 and in base 10; the harmonic language becomes perceptibly poorer, but remains musically interesting.

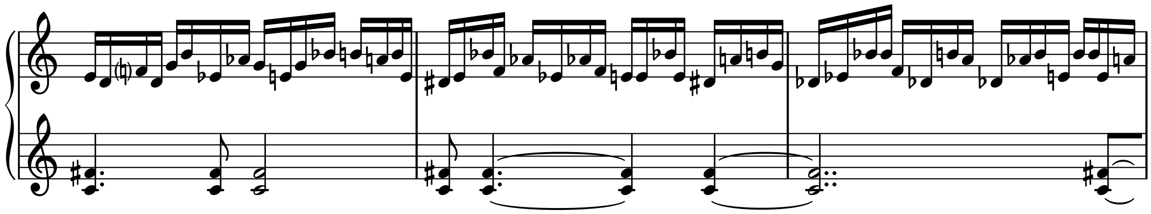

Figure 9: . . . And back to base 10, but with a twist: a second voice provides the two pitches missing from the total chromatic space.

Different numeral bases may be also be combined: for example, an integer sequence in base 10 may provide intervals between notes as seen above (figure 5), while another sequence in base 2 will determine, for each of these notes, whether the melody must go up or down — and other sequences in other bases may even be used at the same time, for example to determine durations and dynamics). Thus, music may be generated procedurally while remaining diverse and surprising.

Furthermore, other numeral bases may be of interest to musicians: for example, when using microtonal resources (translating hexadecimal numbers into 16-ET pitches, and so on), or specific modal scales that are not octave-bound. Even simpler, the binary base remains an useful engine to switch between two possible choices (for example between "up" and "down" as mentioned above, or between major and minor triads, as illustrated in figure 11). Lesser-known is the *phinary* base [3] (built upon the φ "golden ratio" number), which allows for a very surprising syntax — where, for example, "11" may be rewritten as "100" — that is prone to generate some exciting rhythmic structures.

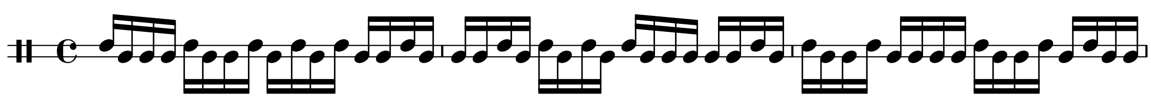

Figure 10: The first "phigits" of π's expansion in the golden base (100.0100101010100100010101010000010100100001001010100). Unlike base 2, this expansion is not evenly distributed between zeroes and ones, which makes performing it as a rhythm more dynamic and captivating.

Translating words into pitches has also been challenging for many centuries; where many composers opted to use only letters from A to G (and H in German, which is evidently convenient when B-A-C-H is involved), some astute tricks had to be found (for example translating S into *Es*, meaning "E flat"). Others opted to simply cycle through the diatonic scale, thus reducing each additional letter to its modulo. Which is why, when several French composers wrote musical tributes to Haydn in 1908, the letters Y and D had to be played as a repeated note (a choice bitterly disputed by Saint-Saens who, alongside with Fauré, refused to have any part in it for that very reason). As a matter of fact, we have yet to find a definite, practical way of translating the whole alphabet into distinct pitches.

[3]This base was first described by George Bergman at a very young age: he was fourteen when his paper was published (Bergman, 1957), but is said to have written it two years earlier.

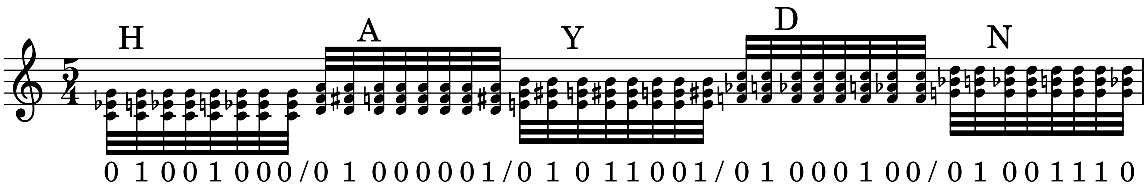

Figure 11: A wholly impractical way of turning words into notes: letters are encoded in UTF-8, then expressed in binary and converted into minor and major triads!

Combinatorial procedures

Combinatorial composition has always been an essential part of Ou*x*po activity — in an article included in *La Littérature Potentielle* (Oulipo, 1973), French mathematician and Oulipo member Claude Berge starts by referring to both Leibniz's *De Arte Combinatoria* (1666), and Euler's *Briefe an eine deutsche Prinzessin* (1769). Another Oulipo member with a specific interest in systematic combinations was Raymond Queneau, who turned the medieval *sextine*'s permutation scheme (based on a set of 6-verse stanzas) into a much more complete and rational tool, opening the way to an entire field of new poetic forms: *terine*, *quatrine*, and ultimately, *quenines* and *pseudo-quenines* of any order.

Such procedures are gladly used by Oumupo, even more happily so since music provides us with ways of taking advantage of these schemes as two-dimensional matrices. Another, more recent example of an interesting combinatorial game is the *Eodermdrome*, a recombination of a 5-element set using specific paths through a non-planar graph. This was initially described by G. Bloom, J. Kennedy and P. Wexler in 1980, then introduced in France by Oulipo member Jacques Roubaud, and expanded upon by the Oulipo list, which in turn brought it to our attention.

Music, compared to literature, is combinatorially agnostic: the constraint of meaning attributed to words that make certain literary combinations valid and others not, tends to lose its relevance in music. For example, the word *note* can have its letters reorganized into *tone*, or the proper noun *Eton*, whereas *neto* (a Spanish word) is not in the English lexicon, and *eotn* does not appear to be a word in any language. Musical notes, on the other hand, can be combined in any order and carry a subjective argument in musical time. Therefore, we Oumupo composers are often prone to present every possible result of a given musical set (an example of which may be seen in figure 12), rather than having to make arbitrary choices based on whether we like some results more than others.

Figure 12: Combinatorics at its most extreme: Tom Johnson's *Chord Catalog* for keyboard (1985) is performed by sequentially playing all of the 8178 chords possible within one octave.

Several classic mathematical tools can be used to articulate music's combinatorial potential: magic squares, symmetry, Pascal's triangle, the sieve of Eratosthenes, prime numbers, and many others. Stochastic processes may also be of use, although the strictest Ou$x$po line of thought tends not to be interested in pure randomness; what follows is an example toeing the line of our field of study.

Attractors

Attractors are a tool used by composers looking for music that contains qualities of both structure and chaos.

One interesting example of attractors in music comes from the Moscow-based composer Sergei Zagny (born in 1960), whose electronic composition *Formula 1* (Zagny, 2000) is based on mathematical equations elaborated by the Belgian mathematician Pierre-François Verhulst (1804-1849). In 1845, Verhulst described a possible model for animal population growth through an equation that came to be known as the "logistic map" (Verhulst, 1845): a variable x (which must always stay between 0 and 1) is recursively multiplied by $(1 - x)$ and then by a factor a (that can be defined between 0 and 4). The resulting curve (illustrated as a bifurcation diagram in figure 13) has both chaotic and fractal properties, which makes it particularly interesting for musical purposes.

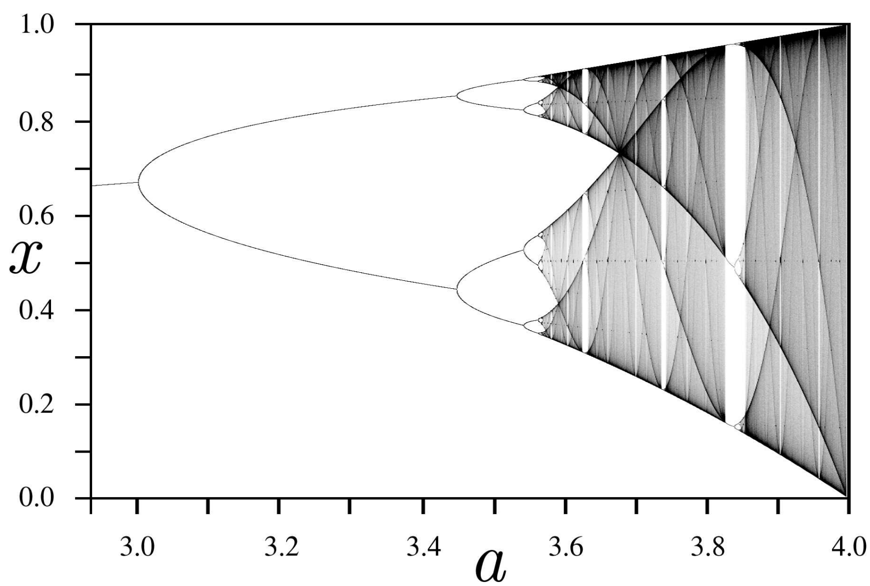

Figure 13: Bifurcation diagram for Verhulst's "logistic map".

Using different initial values for a, we can observe the following iterative results, as detailed in figures 14 to 16.

```
a = 2.6; x = 0.5; ct = 0;
While[ct <= 30, ct++; x = N[a*x*(1 - x)];
Print[ct, "␣␣", x]]

0.652        0.59153      0.6282324    0.6072475    0.6200956    0.6125017
0.6170938    0.6143529    0.61600210   0.61501311   0.61560712   0.61525113
0.61546514   0.61533715   0.61541316   0.61536717   0.61539518   0.61537819
0.61538820   0.61538221   0.61538622   0.61538423   0.61538524   0.61538425
0.61538526   0.61538527   0.61538528   0.61538529   0.61538530   0.61538531
0.615385     ...
```

Figure 14: with $a = 2.6$ and $x = 0.5$ as a starting point, x becomes 0.65 then 0.59, 0.62, 0.61 etc. until it stabilizes at 0.615385.

```
a = 3.56994; x = 0.5; ct = 0;
While[ct <= 30, ct++; x = N[a*x*(1 - x)];
Print[ct, "␣␣", x]]

0.8924852    0.3425553    0.8039914    0.5625865    0.8785026    0.3810437
0.8419688    0.4750099    0.89025510   0.34878611   0.81085612   0.54751813
0.88442414   0.36491215   0.82733816   0.50996617   0.8921318    0.34354919
0.80510320   0.56016621   0.87956222   0.37817323   0.83950124   0.48101125
0.89119826   0.34615727   0.80799328   0.55384229   0.88213630   0.37117431
0.833238     ...
```

Figure 15: $a = 3.56994$, which is one of Feigenbaum's constants. The result is much more chaotic. In spite of the large differences between later smaller significant figures, the system audibly alternates between two poles — one close to 0.8 and the other close to 0.5. The attractor thus becomes seemingly periodic.

```
a = 3.745; x = 0.5; ct = 0;
While[ct <= 100, ct++; x = N[a*x*(1 - x)];
Print[ct, "␣␣", x]]

0.936252     0.2235243    0.6499864    0.8520045    0.472226     0.933367
0.2329368    0.6691459    0.82910610   0.53062711   0.93273712   0.23495613
0.6731714    0.82394615   0.54324616   0.92924617   0.24622518   0.69506619
0.7937520    0.61309721   0.88834822   0.37145123   0.87436424   0.41139425
0.90684826   0.31635927   0.80995328   0.57646429   0.91435430   0.29327331
0.77620432   0.65054933   0.8513734    0.47388935   0.93369736   0.23184237
0.66695238   0.83186539   0.52379640   0.93412941   0.23043642   0.66412143
0.83537644   0.51502445   0.93540546   0.22628347   0.65567148   0.84549549
0.4892250    0.93581551   0.22494552   0.65292153   0.84867454   0.48095855
0.93489256   0.22795457   0.65908658   0.8414759    0.49957660   0.93624961
0.22352662   0.6499963    0.85199864   0.47223465   0.93336366   0.23292767
0.66912668   0.82912969   0.5305770    0.9327571    0.23491372   0.67308573
0.82405574   0.54298175   0.92933276   0.2459577    0.69454378   0.79451379
0.61141780   0.8897681    0.36733582   0.87033883   0.42262184   0.91382785
0.29490886   0.77872687   0.64530888   0.85717689   0.45848390   0.92979591
0.2444692    0.69169993   0.79862694   0.6022895    0.89707396   0.34578797
0.84718898   0.4848399    0.935388100  0.226337101  0.655782     ...
```

Figure 16: $a = 3,745$. After a few iterations, we reach the "period five window", where the attractor loops around five values that begin respectively with 0,8, 0,5, 0,9, 0,2 and 0,6. This configuration is less stable and the numbers obtained are completely dispersed. Nevertheless, if one listens to the result in time, one can hear an emergent logic in a random alteration between five distinct values. In this way, music may reveal more clearly what a purely numerical system often obfuscates (graphical representations also are an efficient tool in this regard, as illustrated in figure 13).

Although not a member of Oumupo, Zagny demonstrates a relevant approach to automated composition, where controlled chaos remains both ordered and reproducible. Thus, attractors illustrate an underlying thema through most of our own musical research, as elaborated below: the tense equilibrium between unpredictability and determinism.

Tracing paths through the *Tonnetz*

The *Tonnetz* (tone-network) is a geometrical structure originally introduced by the Swiss mathematician Leonhard Euler (1707-1783). First in 1739, then in his 1774 treaty *De harmoniæ veris principiis per speculum musicum repræsentatis* ("On the actual principles of harmony as represented in a musical mirror")[4], Euler introduces a representation of musical pitch in a two-dimensional space (see figure 17, left). The space's two axes are the perfect fifth and the major third. After the octave, these are the two most "consonant" intervals, meaning that their relation can be expressed with low integer ratios (respectively 3:2 for the perfect fifth and 5:4 for the major third).

The modern version of the *Tonnetz*, more commonly used in contemporary musical discourse (see figure 17, right), makes use of these two axes and adds a minor third axis so that the two-dimensional plane is now divided into triangles. By construction, one side of every triangle always is a perfect fifth, so that all triangles correspond to major and minor triadic chords. Moving outward from the barycenter of the triangle (corresponding, for example, to C-E-G), one notices three major axes of symmetry that allow for paths through the *Tonnetz* that are achieved by changing one of the three chordal notes by one semitone or one whole tone at most.

- The *relative* chord (R) — C-E-A in the above example.
- The *parallel* chord (P) — C-E♭-G.
- The *leading-tone* chord (L) — B-E-G.

An interesting consequence of this partitioning is that major chords only ever lead to minor chords and, inversely, minor chords only ever lead to major chords.

[4] A comprehensive list of Euler's work may be found on the Euler Archive's website; see bibliography.

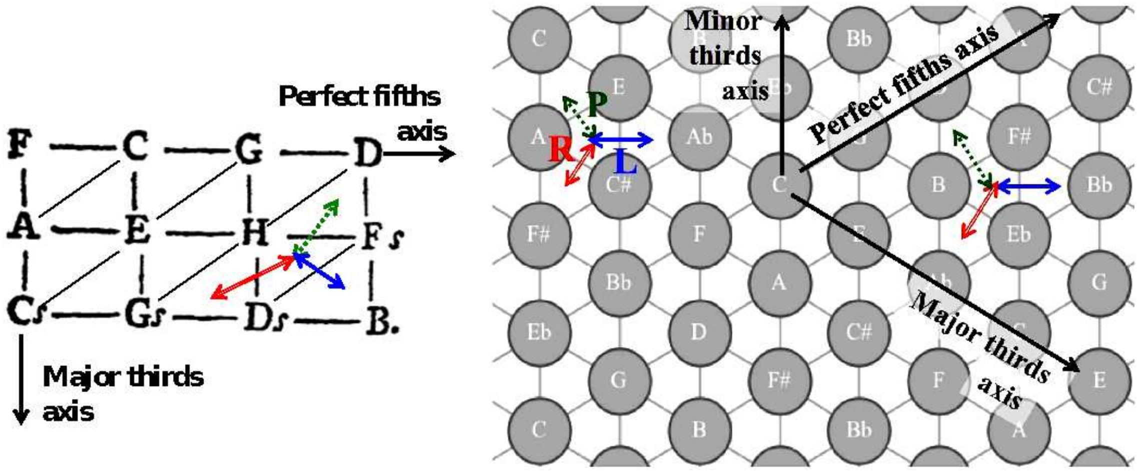

Figure 17: Symmetries in the *Tonnetz*: three transformations of a B major triad (B-D♯-F♯, or H-Ds-Fs in German), in Euler's representation (left) and in the model preferred nowadays (right).

From this triangulation of the musical plane, one can derive its dual graph: a hexagonal tiling (not unlike a bee's nest) where every note finds itself at the center of a hexagon surrounded by six summits — the six notes with which it can form a major or minor chord. By means of the three operators R, P, and L, all of which preserve two out of three notes in a three-note chord and change the moving note by no more than a whole tone, one can navigate through the lattice to create harmonic progressions that only require minimal voice-leading (Andreatta, 2016).

Of the many different paths that can be traced through the *Tonnetz*, some have particularly interesting qualities. For example, from any given chordal point on a hexagon in the *Tonnetz*, it is possible to create a path that exhausts all twenty-four major and minor chords without repetition. We call this path a *Hamiltonian* traversal — the word Hamiltonian coming from an analogous operation in graph theory. If a path is cyclic, one calls it a "Hamiltonian cycle." An exhaustive research has shown that there are 124 of these Hamiltonian cycles through the *Tonnetz*, that can be classified according to their internal symmetries. There are, for example, cycles that zig-zag because they are comprised of only two elemental symmetries (for example, alternating L and R operations). Other cycles traverse the 24 major and minor chords without internal symmetries, i.e., without any recurring sub-pattern. Figure 18 inventories the 28 Hamiltonian cycles in the *Tonnetz* having an internal periodicity. We find here, in positions 16 and 28, two zig-zag configurations that allows one to visit every possible minor and major chord once and once only (the first solution actually being a retrogradation of the second one, and conversely).

1. C-Cm-A♭-A♭m-E-C♯m-A-Am-F-Fm-C♯-B♭m-F♯-F♯m-D-Dm-B♭-Gm-E♭-E♭m-B-Bm-G-Em ⟶ PLPLRL
2. C-Cm-A♭-Fm-C♯-C♯m-A-Am-F-Dm-B♭-B♭m-F♯-F♯m-D-Bm-G-Gm-E♭-E♭m-B-A♭m-E-Em ⟶ PLRLPL
3. C-Cm-E♭-E♭m-F♯-F♯m-A-C♯m-E-Em-G-Gm-B♭-B♭m-C♯-Fm-A♭-A♭m-B-Bm-D-Dm-F-Am ⟶ PRPRPRLR
4. C-Cm-E♭-E♭m-F♯-B♭m-C♯-C♯m-E-Em-G-Gm-B♭-Dm-F-Fm-A♭-A♭m-B-Bm-D-F♯m-A-Am ⟶ PRPRLRPR
5. C-Cm-E♭-E♭m-F♯-B♭m-C♯-Fm-A♭-A♭m-B-Bm-D-F♯m-A-C♯m-E-Em-G-Gm-B♭-Dm-F-Am ⟶ PRPRLRLR
6. C-Cm-E♭-Gm-B♭-B♭m-C♯-C♯m-E-Em-G-Bm-D-Dm-F-Fm-A♭-A♭m-B-E♭m-F♯-F♯m-A-Am ⟶ PRLRPRPR
7. C-Cm-E♭-Gm-B♭-B♭m-C♯-Fm-A♭-A♭m-B-E♭m-F♯-F♯m-A-C♯m-E-Em-G-Bm-D-Dm-F-Am ⟶ PRLR
8. C-Cm-E♭-Gm-B♭-Dm-F-Fm-A♭-A♭m-B-E♭m-F♯-B♭m-C♯-C♯m-E-Em-G-Bm-D-F♯m-A-Am ⟶ PRLRLRPR
9. C-Em-E-A♭m-A♭-Cm-E♭-Gm-G-Bm-B-E♭m-F♯-B♭m-B♭-Dm-D-F♯m-A-C♯m-C♯-Fm-F-Am ⟶ LPLPLR
10. C-Em-E-A♭m-B-E♭m-E♭-Gm-G-Bm-D-F♯m-F♯-B♭m-B♭-Dm-F-Am-A-C♯m-C♯-Fm-A♭-Cm ⟶ LPLRLP
11. C-Em-G-Gm-B♭-B♭m-C♯-C♯m-E-A♭m-B-Bm-D-Dm-F-Fm-A♭-Cm-E♭-E♭m-F♯-F♯m-A-Am ⟶ LRPRPRPR
12. C-Em-G-Gm-B♭-B♭m-C♯-Fm-A♭-Cm-E♭-E♭m-F♯-F♯m-A-C♯m-E-A♭m-B-Bm-D-Dm-F-Am ⟶ LRPRPRLR
13. C-Em-G-Gm-B♭-Dm-F-Fm-A♭-Cm-E♭-E♭m-F♯-B♭m-C♯-C♯m-E-A♭m-B-Bm-D-F♯m-A-Am ⟶ LRPR
14. C-Em-G-Bm-B-E♭m-E♭-Gm-B♭-Dm-D-F♯m-F♯-B♭m-C♯-Fm-F-Am-A-C♯m-E-A♭m-A♭-Cm ⟶ LRLPLP
15. C-Em-G-Bm-D-Dm-F-Fm-A♭-Cm-E♭-Gm-B♭-B♭m-C♯-C♯m-E-A♭m-B-E♭m-F♯-F♯m-A-Am ⟶ LRLRPRPR
16. C-Em-G-Bm-D-F♯m-A-C♯m-E-A♭m-B-E♭m-F♯-B♭m-C♯-Fm-A♭-Cm-E♭-Gm-B♭-Dm-F-Am ⟶ LR
17. C-Am-A-F♯m-F♯-E♭m-E♭-Cm-A♭-Fm-F-Dm-D-Bm-B-A♭m-E-C♯m-C♯-B♭m-B♭-Gm-G-Em ⟶ RPRPRPRL
18. C-Am-A-F♯m-F♯-E♭m-B-A♭m-A♭-Fm-F-Dm-D-Bm-G-Em-E-C♯m-C♯-B♭m-B♭-Gm-E♭-Cm ⟶ RPRPRLRP
19. C-Am-A-F♯m-F♯-E♭m-B-A♭m-E-C♯m-C♯-B♭m-B♭-Gm-E♭-Cm-A♭-Fm-F-Dm-D-Bm-G-Em ⟶ RPRPRLRL
20. C-Am-A-F♯m-D-Bm-B-A♭m-A♭-Fm-F-Dm-B♭-Gm-G-Em-E-C♯m-C♯-B♭m-F♯-E♭m-E♭-Cm ⟶ RPRLRPRP
21. C-Am-A-F♯m-D-Bm-B-A♭m-E-C♯m-C♯-B♭m-F♯-E♭m-E♭-Cm-A♭-Fm-F-Dm-B♭-Gm-G-Em ⟶ RPRL
22. C-Am-A-F♯m-D-Bm-G-Em-E-C♯m-C♯-B♭m-F♯-E♭m-B-A♭m-A♭-Fm-F-Dm-B♭-Gm-E♭-Cm ⟶ RPRLRLRP
23. C-Am-F-Fm-C♯-C♯m-A-F♯m-D-Dm-B♭-B♭m-F♯-E♭m-B-Bm-G-Gm-E♭-Cm-A♭-A♭m-E-Em ⟶ RLPLPL
24. C-Am-F-Dm-D-Bm-B-A♭m-A♭-Fm-C♯-B♭m-B♭-Gm-G-Em-E-C♯m-A-F♯m-F♯-E♭m-E♭-Cm ⟶ RLRPRPRP
25. C-Am-F-Dm-D-Bm-B-A♭m-E-C♯m-A-F♯m-F♯-E♭m-E♭-Cm-A♭-Fm-C♯-B♭m-B♭-Gm-G-Em ⟶ RLRPRPRL
26. C-Am-F-Dm-D-Bm-G-Em-E-C♯m-A-F♯m-F♯-E♭m-B-A♭m-A♭-Fm-C♯-B♭m-B♭-Gm-E♭-Cm ⟶ RLRP
27. C-Am-F-Dm-B♭-Gm-G-Em-E-C♯m-A-F♯m-D-Bm-B-A♭m-A♭-Fm-C♯-B♭m-F♯-E♭m-E♭-Cm ⟶ RLRLRPRP
28. C-Am-F-Dm-B♭-Gm-E♭-Cm-A♭-Fm-C♯-B♭m-F♯-E♭m-B-A♭m-E-C♯m-A-F♯m-D-Bm-G-Em ⟶ RL

Figure 18: The 28 Hamiltonian cycles with an internal periodicity.

It is worth noting that all other paths generated by the repetition of a couple of symmetric operations, such as *PR* or *LP*, will create chord progressions that are shorter than 24. These two non-Hamiltonian cycles as represented in figure 19.

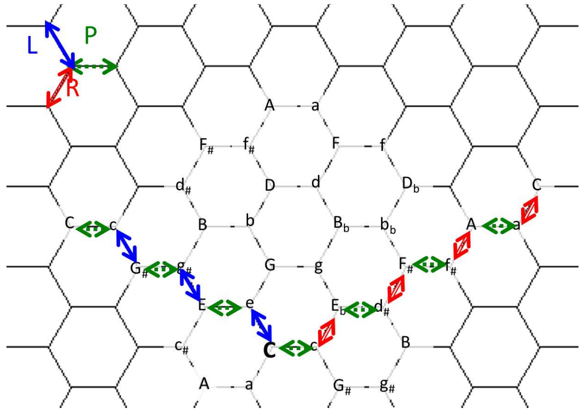

Figure 19: Two "zig-zag"-shaped harmonic cycles that encompass less than 24 chords: *PR* then *LP*, respectively 8- and 6-chord long.

The classification of chord progressions in the *Tonnetz* opens up a wide array of possibilities in the study of musical harmonic organization. From an Oumupo point of view, it gives way to new experiments in composition and writing, particularly in sub-genres that masquerade as "pop" music but actually rely on formal, learned and constraint-based writing. *Hamiltonian Songs* are now the subject of concerts and workshops; in 2016 a Hamiltonian Cabaret, proposed by Fabrice Guedy, Moreno Andreatta and French singer-songwriter Polo, allowed students to dive into this universe for an entire week. They created a song whose harmonic structure (see figure 20) is based on one of the periodic Hamiltonian cycles listed above.

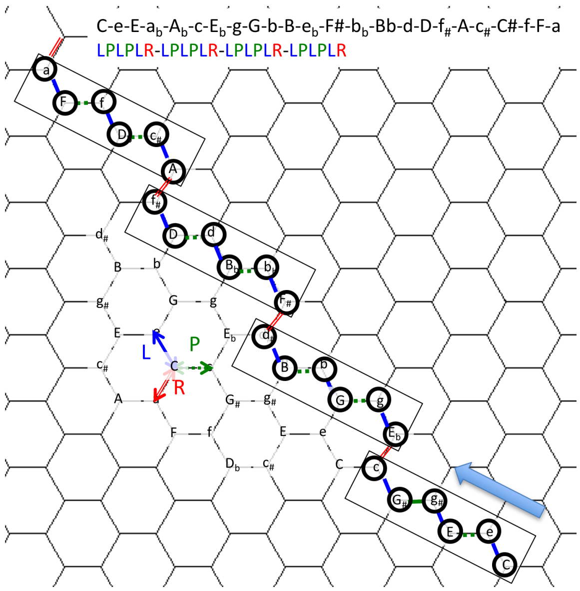

Figure 20: The Hamiltonian cycle underlying a *Ballade-Marabout* written collectively by students of PSL (Paris Sciences & Lettres). The cycle involved here is listed as #9 in figure 18; it is comprised of 4 periodic blocks of 6 chords each.

Putting a limited set of resources in the most efficient and rational order has long been an obsession of all Ou*x*po groups, to begin with Oulipo; one is reminded, for example, of the so-called "Knight's Tour" that Georges Perec used in *Life, A User's Manual* (1978), envisioning his whole "novels" as a 10×10 chessboard.

Hamiltonian graphs demonstrate, once more, the flexibility of music in comparison with words and letters. Using all 24 triadic chords without any repetition (or, in the case of serial music, all 12 tones of a given series) seems far more achievable than, for example, the elusive *heteropangram* which the Oulipo list has spent decades striving to achieve: a sentence of 26 letters where no letter is used more than once... and that still makes sense.

Melodic generation and transformations

Beyond their apparent simplicity, some melodies can prove to be fairly complex objects as soon as one attempts to describe them in rational or procedural terms. This has been one of Tom Johnson's fields of study long before he became a member of Oumupo, from his *Rational melodies* (1982) to more recent works (Johnson, 1996).

Such experiments, needless to say, have existed for many centuries; to name but one, examples of **canons** are found as early as the 14$^{\text{th}}$ century (and probably even predate musical palindromes, which we mentioned previously). Numerous composers, including the well-known J.S. Bach, have proven that a melody can act as a counterpoint to itself, when either shifted by a certain duration, reversed ("crab canon") or played simultaneously at different tempos. More recently, the study of canons has gained new traction following Dan Tudor Vuza's work on their rhythmic structures (Vuza, 1991-1992). A relatively new field in musical composition, the so-called "tiling canon" is of particular interest to Oumupo, and with some help from the Oulipo list, we have even tried to account for the possibility of multi-layered lyrics, whose meaning evolve when additional syllables are interpolated between different voices.

Another example of structures that are both melodic and "potentially" polyphonic, is provided by **self-similar melodies**, i.e., melodies that replicate themselves at different time scales (not unlike fractals in the mathematical realm).

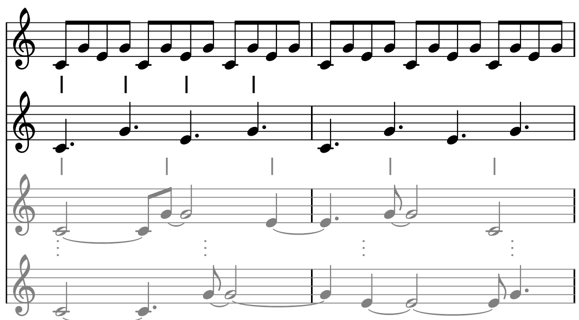

Figure 21: The so-called "Alberti bass" happens to be a self-replicating loop: one can find the exact same melody when playing only every third note, but this property is also verified with other ratios such as 5 or 7.

Concretely, the melody re-articulates itself by only playing one note out of every n notes (see figure 21). In the most complex cases, n can change but the melody remains invariant. This property has notably been explored by Tom Johnson in his 1998 octet *La Vie Est Si Courte* (figure 22).

Figure 22: *La Vie Est Si Courte*, T. Johnson (1998).

The whole point of Ou*x*po schools of thought is to restrict the number of possibilities, thereby limiting the author's reliance on arbitrary choices or, *horrescimus referens*, "inspiration". Which does not exonerate authors of any control nor responsibility: some choices still have to be made (the least of which is indeed not *which* formula to choose in the first place); furthermore, musical composition has always been a point of tension between artistic freedom and formal determinism.

With this in mind, one can notice that where forms such as canons and self-similarity define a set of criteria that allow us to choose between multiple possible melodies (or even to adapt a pre-existing melody by somehow making it fit, more or less artificially, inside the requisite frame), they still do leave a lot of room for many "potential" solutions to the problem at hand. Hence the need for another kind of tools: "generative" constraints, where one can let an algorithmic automaton roam free and generate the *whole melody*.

An example of such a process is as simple as it is fascinating: the **Dragon Curve**, first described by Martin Gardner in a 1967 issue of *Scientific American* (Gardner, 1967), has become a familiar and popular entry into mathematical concepts... but few have applied it to the creation of a musical score.

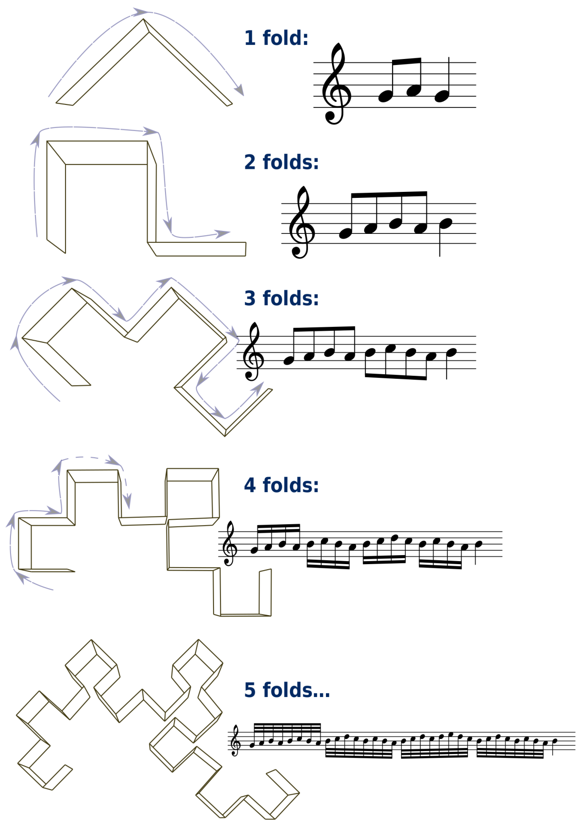

Figure 23: How to tuck your dragon.

We all know the simple game of folding a slip of paper n times in the same direction and then unfolding the paper to see the final form. The complexity of the exercise increases according to the number of folds (as demonstrated in figure 23). This "Dragon Curve" possesses several interesting properties (fractal, self-similar, symmetric but non-periodic, etc.). The curve can also be read as a series of melodic movements: for each "bump" in the paper, go up one note; for each dip in the paper, go down by one note.

One can hardly fail to notice that for every extra fold, the melody becomes more and more complex, but also more and more maze-like and mysterious — even more so if a specific, non-standard scale has been chosen in the first place: for example a modal scale such as C D♯ E F G A♭ B, for a somewhat exotic effect. This goes to illustrate the tension stated above between freedom and determinism: different composers will make different choices, and therefore get different musical outcomes.

This recreational game was the basis of a piece written by Tom Johnson in 1979 for a student orchestra, *Dragons in A*. The simplicity of its procedural engine allows children to understand the experiment, reproduce it and possibly make it their own, which highlights an essential aspect of Oumupo's pedagogical subtext: by demystifying musical composition and turning it into an accessible game that is both logical and playful, we hope to reconcile the broadest, most diverse audience with contemporary artistic creation.

The Dragon Curve has been happily adopted by several other Oumupo contributors in new, unforeseen ways. Figure 24 shows an example where the curve has been given additional dimensions by iterating 120-degree angles instead of right angles, and then running it through a *Tonnetz*.

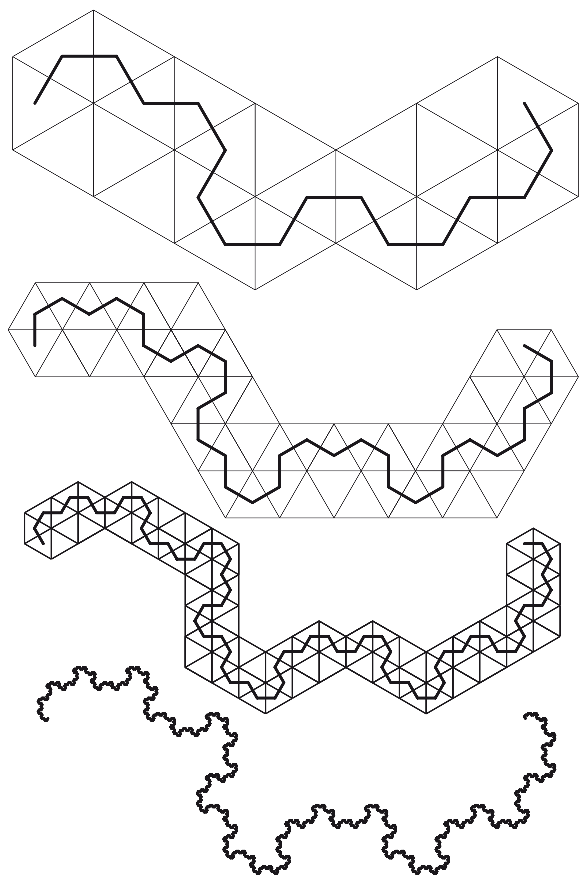

Figure 24: How often do you get to see a dragon in a honeycomb?

Melodies and harmonies generated by the Dragon Curve are yet another case where music, through its breathtaking and hypnotic beauty, allows one to “hear” mathematical constructs, and to comprehend them in exciting new ways.

A potential conclusion?

Starting with the assumption that music is much more flexible than literature with respect to the combinatory potential of its own material, we have presented some of the ideas that Oumupo has set out to explore in recent years[5].

Although mathematics are an integral part of our activity (and the same could be said of any musician), we have also made it a point to engage as much as possible with other languages, especially by exchanging with Ou*x*po groups that apply similar tools and methods to other expressive media — to begin with Oulipo, our historical common reference.

Much like Oulipo does not define itself as a literary movement, all Oumupo members pursue a creative career of their own by taking inspiration from our collective reflection on constraint-based writing, and in turn influence ongoing projects with their own research field and interests. Far from being felt as limiting, these constraints are a source of freedom and artistic courage to all of us. To paraphrase Georges Perec: "At the end of the day, I give myself constraints in order to be entirely liberated."

Acknowledgements

Our most sincere thanks go to Oumupo member **Mike Solomon** for translating an early draft of this paper, as well as to Oumupo contributor **Gilles Esposito-Farese**, who also happens to be a long-time, valuable member of the Oulipo mailing list.

All musical examples were typeset using GNU LilyPond (`http://lilypond.org`), an integral tool in many of our experiments and daily creative activities.

References

Dartmouth University, Euler Archive, `http://www.eulerarchive.org/`, 2004-2017.

Pierre-François Verhulst, 'Recherches mathématiques sur la loi d'accroissement de la population', *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles*, 1845.

George Bergman, 'A Number System with an Irrational Base', *Mathematics Magazine* vol. 31, 1957, pp. 98-110.

Martin Gardner, 'Mathematical Games', *Scientific American* vol. 216, 1967, p. 118.

Oulipo (col.), *La Littérature potentielle* Gallimard, Paris, 1973, pp. 19-27 and pp. 47-50.

Oulipo (col., edited by Marcel Bénabou and Paul Fournel), *Anthologie de l'Oulipo*, Gallimard, Paris, 2009, pp. 798-801.

Dan Tudor Vuza, 'Supplementary Sets and Regular Complementary Unending Canons', *Perspectives of New Music* vol. 29 to 31, 1991-1992.

[5] Some of our theoretical work and musical examples may be found on Oumupo's website, mostly in French: `http://oumupo.org`.

Tom Johnson, *Self-similar melodies*, Editions 75, Paris, 1996.

Sergei Zagny, 'Formula 1' in *Texts on Music*, `http://conceptualism.letov.ru/sergei-zagny/`, 1995-2000, p. 18.

Moreno Andreatta and Gilles Baroin, 'An Introduction on Formal and Computational Models in Popular Music Analysis and Generation' in *Aesthetics and Neuroscience: Scientific and Artistic Perspectives*, Springer International Publishing, Cham, 2016, p. 257.

Moreno Andreatta, Martin Granger, Tom Johnson and Valentin Villenave, 'L'Oumupo, entre musique et mathématiques' in *Maths Langages Express*, `http://www.cijm.org/`, 2017, p. 15.

Valentin Villenave *et al.*, Oumupo website, `http://oumupo.org/`, 2011-2017.